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Canonical Correlations Associated With Symmetric Reflexive Generalized Inverses of the Dispersion Matrix

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ABSTRACT

We consider the relations between two sets of canonical correlations: one based on a (possibly singular) dispersion matrix $\mathbf{V}$, and the other on a symmetric reflexive generalized inverse of $\mathbf{V}$. Special attention is paid to the number of unit canonical correlations in each set. We establish series of results characterizing the situations where all canonical correlations in one or both sets are less than one. It turns out that in all such situations the canonical correlations in the one set are uniformly comparable to those in the other set.

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1. INTRODUCTION

Consider two random vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ whose joint positive definite dispersion matrix is partitioned as

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 \end{pmatrix} = \mathbf{X}'\mathbf{X}, \tag{1.1}$$

where $\mathbf{A}'$ denotes the transpose of a matrix $\mathbf{A}$, and $\mathbf{X} = (\mathbf{X}_1 : \mathbf{X}_2)$ denotes the partitioned $n \times (p+q)$ matrix comprising of $\mathbf{X}_1$ $(n \times p)$ and $\mathbf{X}_2$ $(n \times q)$. The canonical correlations based on (1.1) offer a wide area for application of matrix theory; see, for example, Rao (1973, Section 8f) and Anderson (1984, Section 12.2). Among the first authors who have studied the canonical correlations under a singular $\mathbf{V}$, we may mention Khatri (1976), Seshadri and Styan (1980), Rao (1981), Yanai (1981), and Styan (1985).

One specific question to ask is what is the relation between the canonical correlations based on (1.1) and the canonical correlations based on the inverse of $\mathbf{V}$, this inverse being partitioned according to (1.1). Jewell and Bloomfield (1983) showed that these two sets of canonical correlations are precisely the same. A natural generalization is to consider a singular $\mathbf{V}$ and partition its generalized inverse according to (1.1) and to study the properties of the resulting canonical correlations. We cannot, however, take any generalized inverse of $\mathbf{V}$: it must be nonnegative definite (and symmetric).

Puntanen (1987, pp. 43–46) and Latour, Puntanen, and Styan (1987) considered the matrix

$$\mathbf{V}^{\#} = \begin{pmatrix} \mathbf{V}_{11\cdot 2}^{\sim} & -\mathbf{V}_{11\cdot 2}^{\sim}\mathbf{V}_{12}\mathbf{V}_{22}^{\sim} \\ -\mathbf{V}_{22}^{\sim}\mathbf{V}_{21}\mathbf{V}_{11\cdot 2}^{\sim} & \mathbf{V}_{22}^{\sim} + \mathbf{V}_{22}^{\sim}\mathbf{V}_{21}\mathbf{V}_{11\cdot 2}^{\sim}\mathbf{V}_{12}\mathbf{V}_{22}^{\sim} \end{pmatrix}, \tag{1.2}$$

where $\mathbf{V}_{11\cdot 2} = \mathbf{X}_1'\mathbf{Q}_2\mathbf{X}_1$, with $\mathbf{A}^{\sim}$ denoting a symmetric reflexive generalized inverse of $\mathbf{A}$, and $\mathbf{Q}_2$ denoting the orthogonal projector onto the orthocomplement of the column space of $\mathbf{X}_2$. The matrix $\mathbf{V}^{\#}$ is a symmetric reflexive generalized inverse of $\mathbf{V}$ for any choices of symmetric reflexive generalized inverses $(\mathbf{X}_2'\mathbf{X}_2)^{\sim}$ and $(\mathbf{X}_1'\mathbf{Q}_2\mathbf{X}_1)^{\sim}$; cf. Marsaglia and Styan (1974a, p. 439). Following Latour, Puntanen, and Styan (1987), we may say that $\mathbf{V}^{\#}$, defined in (1.2), is in Banachiewicz-Schur form.

The numbers of unit canonical correlations associated with $\mathbf{V}$ and $\mathbf{V}^{\#}$ are worth special attention; let us denote them as u and $u^{\#}$, respectively. While studying the properties of $\mathbf{V}^{\#}$, Puntanen (1987, p. 45) showed that $u^{\#} = 0$, and that if $u = 0$, then $\mathbf{V}^{\#}$ and $\mathbf{V}$ possess precisely the same canonical correlations. Latour, Puntanen, and Styan (1987) proved that if $\mathbf{V}^{\#}$ and $\mathbf{V}$ possess the same canonical correlations for *all* choices of $\mathbf{V}^{\#}$ of the form (1.2), then $u = 0$. They further introduced inequalities between these two sets of canonical correlations. Related studies were done later by Khatri (1990), who partitioned $\mathbf{V}^{\#}$ as $\mathbf{V}^{\#} = (\mathbf{V}^{\#}_{(1)} : \mathbf{V}^{\#}_{(2)})$ and studied the canonical correlations between the random vectors $(\mathbf{V}^{\#}_{(1)})'\mathbf{x}_1$ and $(\mathbf{V}^{\#}_{(2)})'\mathbf{x}_2$, whose joint dispersion matrix is then $\mathbf{V}^{\#}$.

The purpose of this paper is to study the properties of the canonical correlations based on *any* symmetric reflexive generalized inverse of $\mathbf{V}$, and thereby extend the results concerning a generalized inverse being in Banachiewicz-Schur form. Our approach is based on extensive use of geometrical concepts associated with the projection operator. The statistical considerations utilize several algebraic results which seem to be of interest independently of the statistical context, and are therefore established in Section 2.

Regarding the notation, we denote by $\mathbb{R}^n$ the vector space of real n-tuples, and by $\mathbb{R}^{n\times m}$ the set of $n \times m$ real matrices. The symbols $\mathbf{A}^-, \mathbf{A}^+, \mathscr{C}(\mathbf{A})$, and $r(\mathbf{A})$ stand for a generalized inverse, the Moore-Penrose inverse, the column space, and the rank, respectively, of $\mathbf{A}$. Moreover, $\mathscr{C}^{\perp}(\mathbf{A})$ denotes the orthocomplement of $\mathscr{C}(\mathbf{A})$, and $\mathbf{P_A}$ and $\mathbf{Q_A}$ denote the orthogonal projectors on $\mathscr{C}(\mathbf{A})$ and $\mathscr{C}^{\perp}(\mathbf{A})$, respectively, i.e., $\mathbf{P_A} = \mathbf{AA}^+$ and $\mathbf{Q_A} = \mathbf{I}_n - \mathbf{AA}^+$, where $\mathbf{I}_n$ is the identity matrix of order n. Furthermore, $\mathbf{P}_{(\mathbf{A}:\mathbf{B})}$ denotes the orthogonal projector onto the column space of the partitioned matrix $(\mathbf{A} : \mathbf{B})$, and $\mathbf{Q}_{(\mathbf{A}:\mathbf{B})} = \mathbf{I}_n - \mathbf{P}_{(\mathbf{A}:\mathbf{B})}$. The notation $\mathscr{C}(\mathbf{A}) \boxplus \mathscr{C}(\mathbf{B})$ refers to the orthogonal direct sum of $\mathscr{C}(\mathbf{A})$ and $\mathscr{C}(\mathbf{B})$.

2. PRELIMINARY ALGEBRAIC RESULTS

For $\mathbf{A} \in \mathbb{R}^{n\times p}$ and $\mathbf{B} \in \mathbb{R}^{n\times q}$ define

$$\mathbf{P}_{\mathbf{A}\cdot\mathbf{B}} = \mathbf{A}(\mathbf{A}'\mathbf{Q_B}\mathbf{A})^{-}\mathbf{A}'\mathbf{Q_B}. \tag{2.1}$$

Being idempotent, this matrix is a projector. More precisely, it is the projector on $\mathscr{C}[\mathbf{A}(\mathbf{A}'\mathbf{Q_B}\mathbf{A})^{-}\mathbf{A}'\mathbf{Q_B}]$ along $\mathscr{C}^{\perp}(\mathbf{Q_B}\mathbf{A})$. The first of these subspaces is not invariant with respect to the choice of $(\mathbf{A}'\mathbf{Q_B}\mathbf{A})^{-}$ except for the situations characterized in the following lemma.

LEMMA 1. *For any* $\mathbf{A} \in \mathbb{R}^{n \times p}$ *and* $\mathbf{B} \in \mathbb{R}^{n \times q}$, *the following seven statements are equivalent*:

(a) $\mathbf{P}_{\mathbf{A}\cdot\mathbf{B}}$ *is invariant with respect to the choice of* $(\mathbf{A}'\mathbf{Q}_{\mathbf{B}}\mathbf{A})^{-}$;
(b) $\mathscr{C}[\mathbf{A}(\mathbf{A}'\mathbf{Q}_{\mathbf{B}}\mathbf{A})^{-}\mathbf{A}'\mathbf{Q}_{\mathbf{B}}]$ *is invariant with respect to the choice of* $(\mathbf{A}'\mathbf{Q}_{\mathbf{B}}\mathbf{A})^{-}$;
(c) $\mathrm{r}(\mathbf{Q}_{\mathbf{B}}\mathbf{A}) = \mathrm{r}(\mathbf{A})$;
(d) $\mathbf{P}_{\mathbf{A}\cdot\mathbf{B}}\mathbf{A} = \mathbf{A}$;
(e) $\mathbf{P}_{\mathbf{A}\cdot\mathbf{B}}$ *is the projector onto* $\mathscr{C}(\mathbf{A})$ *along* $\mathscr{C}(\mathbf{B}) \boxplus \mathscr{C}^{\perp}(\mathbf{A} : \mathbf{B})$;
(f) $\mathscr{C}(\mathbf{A}) \cap \mathscr{C}(\mathbf{B}) = \{\mathbf{0}\}$;
(g) $\mathbf{P}_{(\mathbf{A}:\mathbf{B})} = \mathbf{P}_{\mathbf{A}\cdot\mathbf{B}} + \mathbf{P}_{\mathbf{B}\cdot\mathbf{A}}$.

Proof. Since $\mathscr{C}(\mathbf{A}'\mathbf{Q}_{\mathbf{B}}\mathbf{A}) = \mathscr{C}(\mathbf{A}'\mathbf{Q}_{\mathbf{B}})$, it follows from Rao and Mitra (1971, pp. 21, 43) and from Baksalary and Kala (1983, Theorem) that conditions (a) and (b) are equivalent; they hold if and only if $\mathscr{C}(\mathbf{A}') \subseteq \mathscr{C}(\mathbf{A}'\mathbf{Q}_{\mathbf{B}})$, which is equivalent to (c). The part "(c) $\Rightarrow$ (d)" follows from Rao and Mitra (1971, Lemma 2.2.6), and the part "(d) $\Rightarrow$ (e)" is a consequence of $\mathbf{P}_{\mathbf{A}\cdot\mathbf{B}}\mathbf{B} = \mathbf{0}$ and $\mathbf{P}_{\mathbf{A}\cdot\mathbf{B}}\mathbf{Q}_{(\mathbf{A}:\mathbf{B})} = \mathbf{0}$. Now suppose that $\mathbf{Aa} = \mathbf{Bb}$ for some $\mathbf{a} \in \mathbb{R}^{p}$ and $\mathbf{b} \in \mathbb{R}^{q}$. If (e) holds, then premultiplying this equality by $\mathbf{P}_{\mathbf{A}\cdot\mathbf{B}}$ yields $\mathbf{Aa} = \mathbf{0}$, thus establishing that (e) implies (f). The part "(f) $\Rightarrow$ (c)" is an immediate consequence of the equality

$$\mathrm{r}(\mathbf{A}'\mathbf{Q}_{\mathbf{B}}) = \mathrm{r}(\mathbf{A}) - \dim \mathscr{C}(\mathbf{A}) \cap \mathscr{C}(\mathbf{B});$$

see, e.g., Marsaglia and Styan (1974b, Corollary 6.2). Finally, in view of the symmetry of condition (f), each of conditions (a) through (e) may be supplemented by its counterpart with the roles of $\mathbf{A}$ and $\mathbf{B}$ interchanged, which in particular shows that (g) is equivalent to (d). ∎

If $\mathbf{A}$ is of full column rank and condition (f) above holds, then the generalized inverse in (2.1) becomes the usual inverse. In such a case, the characterization (e) was originally given by Afriat (1957, Theorem 4.1); cf. Rao and Yanai (1979, Note 6) and Yanai (1981, Lemma 1).

LEMMA 2. *Let* $\mathbf{A}, \mathbf{C} \in \mathbb{R}^{n \times p}$ *and* $\mathbf{B}, \mathbf{D} \in \mathbb{R}^{n \times q}$ *be such that*

$$(\mathbf{AC}' + \mathbf{BD}')(\mathbf{A} : \mathbf{B}) = (\mathbf{A} : \mathbf{B}), \tag{2.2}$$

i.e., $(\mathbf{C} : \mathbf{D})'$ *is a generalized inverse of* $(\mathbf{A} : \mathbf{B})$. *Then the following three statements are equivalent*:

(a) $\mathscr{C}(\mathbf{A}) \cap \mathscr{C}(\mathbf{B}) = \{\mathbf{0}\}$,
(b) *either of the conditions* $\mathbf{AC}'\mathbf{A} = \mathbf{A}$ *and* $\mathbf{BD}'\mathbf{A} = \mathbf{0}$ *holds along with either of the conditions* $\mathbf{AC}'\mathbf{B} = \mathbf{0}$ *and* $\mathbf{BD}'\mathbf{B} = \mathbf{B}$,

(c) *the four conditions*

$$\mathbf{AC'A} = \mathbf{A}, \qquad \mathbf{BD'A} = \mathbf{0}, \qquad \mathbf{AC'B} = \mathbf{0}, \quad \textit{and} \quad \mathbf{BD'B} = \mathbf{B} \tag{2.3}$$

hold simultaneously.

If in addition

$$\mathscr{C}(\mathbf{CA}') \subseteq \mathscr{C}(\mathbf{A}:\mathbf{B}) \quad \textit{and} \quad \mathscr{C}(\mathbf{DB}') \subseteq \mathscr{C}(\mathbf{A}:\mathbf{B}), \tag{2.4}$$

then conditions (a), (b), (c) *are also equivalent to*

(d) $\mathbf{AC}' = \mathbf{P}_{\mathbf{A}\cdot\mathbf{B}}$ *and* $\mathbf{BD}' = \mathbf{P}_{\mathbf{B}\cdot\mathbf{A}}$.

Proof. The equivalence of (b) and (c) is an immediate consequence of (2.2). From (2.2) it is also clear that (a) implies (c). Now, let $\mathbf{Aa} = \mathbf{Bb}$ for some $\mathbf{a} \in \mathbb{R}^p$ and $\mathbf{b} \in \mathbb{R}^q$. Then premultipling by $\mathbf{AC}'$ and using the equalities in (c), we get $\mathbf{Aa} = \mathbf{0}$, which completes the proof that (a) is equivalent to (c).

To establish the equivalence between (a), (b), (c), and (d) under the assumption (2.4), first notice that according to Rao and Yania [1979, Theorem 2(d)], a general representation of $\mathbf{AC}'$ satisfying $\mathbf{AC'A} = \mathbf{A}$ and $\mathbf{AC'B} = \mathbf{0}$ is

$$\mathbf{AC}' = \mathbf{P}_{\mathbf{A}\cdot\mathbf{B}} + \mathbf{T}(\mathbf{Q}_{\mathbf{B}} - \mathbf{P}_{\mathbf{Q}_{\mathbf{B}}\mathbf{A}}) = \mathbf{P}_{\mathbf{A}\cdot\mathbf{B}} + \mathbf{TQ}_{(\mathbf{A}:\mathbf{B})}, \tag{2.5}$$

where the second equality is a consequence of $\mathbf{P}_{(\mathbf{A}:\mathbf{B})} = \mathbf{P}_{\mathbf{B}} + \mathbf{P}_{\mathbf{Q}_{\mathbf{B}}\mathbf{A}}$, which results from

$$\mathscr{C}(\mathbf{A}:\mathbf{B}) = \mathscr{C}(\mathbf{B}) \boxplus \mathscr{C}(\mathbf{Q}_{\mathbf{B}}\mathbf{A}); \tag{2.6}$$

cf. (5.14) in Rao and Yanai (1979). But the first condition in (2.4) forces (2.5) to be reduced to the first equality in (d). Similarly, it follows that (a) implies the second equality in (d). Since (d) directly implies that $\mathbf{AC'B} = \mathbf{0}$ and $\mathbf{BD'A} = \mathbf{0}$, which is (b), the proof is complete. ■

Notice that the part "(a) ⇔ (c)" of Lemma 2 is a particular case of a result originally given by Rao and Yanai (1979, Theorem 1). Also notice that conditions (a), (b), (c) are not sufficient for (d) when the assumption (2.4) is not fulfilled. An example is provided by the matrices

$$\mathbf{A} = (1 \quad 0 \quad 0)', \qquad \mathbf{B} = (0 \quad 1 \quad 0)', \qquad \mathbf{C} = (1 \quad 0 \quad 1)', \quad \text{and}$$
$$\mathbf{D} = (0 \quad 1 \quad 1)'.$$

Let $\mathbf{X}_1 \in \mathbb{R}^{n \times p}$ and $\mathbf{X}_2 \in \mathbb{R}^{n \times q}$, denote

$$\mathbf{X} = (\mathbf{X}_1 : \mathbf{X}_2),$$

and consider the nonnegative definite symmetric matrix of the form

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_{11} & \mathbf{V}_{11} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 \end{pmatrix} = \mathbf{X}'\mathbf{X}. \tag{2.7}$$

Now, for a generalized inverse of $\mathbf{V}$ given in the form

$$\mathbf{V}^- = \begin{pmatrix} \mathbf{V}^{11} & \mathbf{V}^{12} \\ \mathbf{V}^{21} & \mathbf{V}^{22} \end{pmatrix} = (\mathbf{V}^{(1)} : \mathbf{V}^{(2)}), \tag{2.8}$$

following Khatri (1990), we define

$$\mathbf{F} = (\mathbf{X}_1 : \mathbf{X}_2)\mathbf{V}^{(1)} = \mathbf{X}_1\mathbf{V}^{11} + \mathbf{X}_2\mathbf{V}^{21} \tag{2.9}$$

and

$$\mathbf{G} = (\mathbf{X}_1 : \mathbf{X}_2)\mathbf{V}^{(2)} = \mathbf{X}_1\mathbf{V}^{12} + \mathbf{X}_2\mathbf{V}^{22}, \tag{2.10}$$

which yields

$$\mathbf{V}\mathbf{V}^- = \begin{pmatrix} \mathbf{X}_1'\mathbf{F} & \mathbf{X}_1'\mathbf{G} \\ \mathbf{X}_2'\mathbf{F} & \mathbf{X}_2'\mathbf{G} \end{pmatrix}. \tag{2.11}$$

The matrices $\mathbf{F}$ and $\mathbf{G}$ have some interesting properties.

LEMMA 3. *For any* $\mathbf{X}_1 \in \mathbb{R}^{n \times p}$ *and* $\mathbf{X}_2 \in \mathbb{R}^{n \times q}$, *let* $\mathbf{F} = \mathbf{X}\mathbf{V}^{(1)}$ *and* $\mathbf{G} = \mathbf{X}\mathbf{V}^{(2)}$, *where* $(\mathbf{V}^{(1)} : \mathbf{V}^{(2)})$ *is a generalized inverse of* $\mathbf{V}$ *defined in* (2.7) *and* $\mathbf{X} = (\mathbf{X}_1 : \mathbf{X}_2)$. *Then:*

(a) $r(\mathbf{F} : \mathbf{G}) = r(\mathbf{X})$,
(b) $\mathscr{C}(\mathbf{F} : \mathbf{G}) = \mathscr{C}(\mathbf{X})$,
(c) $\mathbf{P_X} = \mathbf{X}_1\mathbf{F}' + \mathbf{X}_2\mathbf{G}' = \mathbf{F}\mathbf{X}_1' + \mathbf{G}\mathbf{X}_2' = \mathbf{P}_{(\mathbf{F}:\mathbf{G})}$,
(d) $\mathbf{X}_1'\mathbf{G} = \mathbf{0}$ *if and only if* $\mathscr{C}(\mathbf{G}) = \mathscr{C}(\mathbf{Q}_1\mathbf{X}_2)$, *and* $\mathbf{X}_2'\mathbf{F} = \mathbf{0}$ *if and only if* $\mathscr{C}(\mathbf{F}) = \mathscr{C}(\mathbf{Q}_2\mathbf{X}_1)$.

Proof. Property (a) is established by noting that

$$r(\mathbf{F} : \mathbf{G}) = r[\mathbf{X}(\mathbf{V}^{(1)} : \mathbf{V}^{(2)})] = r(\mathbf{V}\mathbf{V}^-) = r(\mathbf{V}) = r(\mathbf{X}).$$

Property (b) is obtained by combining (a) with the obvious inclusion $\mathscr{C}(\mathbf{F} : \mathbf{G}) \subseteq \mathscr{C}(\mathbf{X})$. Further, (c) follows by way of the equalities

$$\mathbf{P_X} = \mathbf{P}_\mathbf{X}' = \mathbf{P}_{(\mathbf{F}:\mathbf{G})}$$

and

$$\mathbf{P_X} = \mathbf{X}\mathbf{V}^-\mathbf{X}' = (\mathbf{F} : \mathbf{G})\mathbf{X}' = \mathbf{F}\mathbf{X}_1' + \mathbf{G}\mathbf{X}_2'.$$

Finally, if $\mathbf{X}_1\mathbf{G}' = \mathbf{0}$, then from (c) it is seen that $\mathbf{X}_1\mathbf{F}'\mathbf{X}_1 = \mathbf{X}_1$ and $\mathbf{G}\mathbf{X}_2'\mathbf{G} = \mathbf{G}$. Hence $r(\mathbf{X}_1'\mathbf{F}) = (\mathbf{X}_1)$ and $r(\mathbf{X}_2'\mathbf{G}) = r(\mathbf{G})$. On the other hand, from (2.11) it is clear that if $\mathbf{X}_1'\mathbf{G} = \mathbf{0}$, then

$$r(\mathbf{X}) = r(\mathbf{X}_1'\mathbf{F}) + r(\mathbf{X}_2'\mathbf{G}),$$

and on account of $r(\mathbf{X}) = r(\mathbf{X}_1) + r(\mathbf{Q}_1\mathbf{X}_2)$ it follows that

$$r(\mathbf{G}) = r(\mathbf{Q}_1\mathbf{X}_2). \tag{2.12}$$

Since $\mathscr{C}(\mathbf{G}) \subseteq \mathscr{C}(\mathbf{X})$, reexpressing $\mathbf{X}_1'\mathbf{G} = \mathbf{0}$ as $\mathbf{G} = \mathbf{Q}_1\mathbf{G}$ shows that

$$\mathscr{C}(\mathbf{G}) \subseteq \mathscr{C}(\mathbf{Q}_1\mathbf{X}_2), \tag{2.13}$$

and combining (2.13) with (2.12) yields $\mathscr{C}(\mathbf{G}) = \mathscr{C}(\mathbf{Q}_1\mathbf{X}_2)$. The proof of the second part of (d) follows similarly. ■

LEMMA 4. *For any matrices* $\mathbf{A} \in \mathbb{R}^{n\times a}$, $\mathbf{B} \in \mathbb{R}^{n\times b}$, $\mathbf{C} \in \mathbb{R}^{n\times c}$, *and* $\mathbf{D} \in \mathbb{R}^{n\times d}$ *such that* $\mathscr{C}(\mathbf{A}) \subseteq \mathscr{C}(\mathbf{C})$ *and* $\mathscr{C}(\mathbf{B}) \subseteq \mathscr{C}(\mathbf{D})$,

$$\mathrm{ch}_i(\mathbf{P_A P_B}) \leqslant \mathrm{ch}_i(\mathbf{P_C P_D}), \qquad i = 1, \dots, n, \tag{2.14}$$

where $\mathrm{ch}_i(\cdot)$ *denotes the* ith *largest eigenvalue (including multiplicities) of a matrix having real eigenvalues.*

Proof. The inclusion $\mathscr{C}(\mathbf{A})| \subseteq \mathscr{C}(\mathbf{C})$ is equivalent to the difference $\mathbf{P_C} - \mathbf{P_A}$ being nonnegative definite, which is denoted by $\mathbf{P_A} \overset{\mathrm{L}}{\leqslant} \mathbf{P_C}$. Hence $\mathbf{P_B P_A P_B} \overset{\mathrm{L}}{\leqslant} \mathbf{P_B P_C P_B}$, and thus

$$\mathrm{ch}_i(\mathbf{P_A P_B}) = \mathrm{ch}_i(\mathbf{P_B P_A P_B}) \leqslant \mathrm{ch}_i(\mathbf{P_B P_C P_B}) = \mathrm{ch}_i(\mathbf{P_B P_C}). \tag{2.15}$$

Similarly,

$$\mathrm{ch}_i(\mathbf{P_B P_C}) = \mathrm{ch}_i(\mathbf{P_C P_B P_C}) \leqslant \mathrm{ch}_i(\mathbf{P_C P_D P_C}) = \mathrm{ch}_i(\mathbf{P_C P_D}), \tag{2.16}$$

and combining (2.15) with (2.16) yields (2.14). ∎

3. CANONICAL CORRELATIONS

Let $\mathbf{y}$ be an $n \times 1$ random vector with dispersion matrix proportional to the identity matrix, and let $\mathbf{X}_1 \in \mathbb{R}^{n\times p}$ and $\mathbf{X}_2 \in \mathbb{R}^{n\times q}$. Then the joint dispersion matrix of $\mathbf{X}_1'\mathbf{y}$ and $\mathbf{X}_2'\mathbf{y}$ is proportional to the matrix $\mathbf{V}$ given in (2.7). The following lemma comprises three known results on the canonical correlations between $\mathbf{X}_1'\mathbf{y}$ and $\mathbf{X}_2'\mathbf{y}$, which are crucial for further considerations of this paper. They are quoted from Seshadri and Styan (1980, pp. 336, 340) and Styan (1985, Theorem 2.5).

LEAMMA 5. *Let* $\mathbf{X}_1 \in \mathbb{R}^{n\times p}$ *and* $\mathbf{X}_2 \in \mathbb{R}^{n\times q}$, *and let* $\mathbf{y}$ *be an* $n \times 1$ *random vector with the dispersion matrix proportional to* $\mathbf{I}_n$. *Then*

(a) *the canonical correlations between* $\mathbf{X}_1'\mathbf{y}$ *and* $\mathbf{X}_2'\mathbf{y}$ *are equal to the square roots of the nonzero eigenvalues of* $\mathbf{P}_1\mathbf{P}_2$, *i.e,* $\{\mathrm{cc}^2(\mathbf{X}_1'\mathbf{y}, \mathbf{X}_2'\mathbf{y})\} = \{\mathrm{nzch}(\mathbf{P}_1\mathbf{P}_2)\}$;

(b) *the number of unit canonical correlations between* $\mathbf{X}_1'\mathbf{y}$ *and* $\mathbf{X}_2'\mathbf{y}$ *is*

$$u = \dim \mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2);$$

(c) *the canonical correlations between* $\mathbf{X}_1'\mathbf{Q}_2\mathbf{y}$ *and* $\mathbf{X}_2'\mathbf{Q}_1\mathbf{y}$ *are all less than one and are identical with those canonical correlations between* $\mathbf{X}_1'\mathbf{y}$ *and* $\mathbf{X}_2'\mathbf{y}$ *which are not equal to one.*

From (2.9) and (2.10) it is clear that the joint dispersion matrix of $\mathbf{F}'\mathbf{y}$ and $\mathbf{G}'\mathbf{y}$ is

$$\begin{pmatrix} \mathbf{F}'\mathbf{F} & \mathbf{F}'\mathbf{G} \\ \mathbf{G}'\mathbf{F} & \mathbf{G}'\mathbf{G} \end{pmatrix} = (\mathbf{V}^-)'\mathbf{V}\mathbf{V}^-.$$

Notice that $\mathbf{V}^\sim$ is a symmetric reflexive (and thus nonnegative definite) generalized inverse of $\mathbf{V}$ if and only it it admits the representation $(\mathbf{V}^-)'\mathbf{V}\mathbf{V}^-$ for some generalized inverse $\mathbf{V}^-$. Hence it follows that considering the canonical correlations associated with a symmetric reflexive generalized inverse of $\mathbf{V}$ is equivalent to considering the canonical correlations between $\mathbf{F}'\mathbf{y}$ and $\mathbf{G}'\mathbf{y}$, where $\mathbf{F}$ and $\mathbf{G}$ are defined in (2.9) and (2.10), with suitably chosen $\mathbf{V}^- = (\mathbf{V}^{(1)} : \mathbf{V}^{(2)})$.

In this section, we derive a series of results characterizing the situations where all the canonical correlations between $\mathbf{X}_1'\mathbf{y}$ and $\mathbf{X}_2'\mathbf{y}$ and/or between $\mathbf{F}'\mathbf{y}$ and $\mathbf{G}'\mathbf{y}$ are less than one. It turns out that in all such situations the canonical correlations in the one set are uniformly comparable to the canonical correlations in the other set. Lemma 4 is useful for establishing these comparisons.

THEOREM 1. *Let* $\mathbf{X}_1 \in \mathbb{R}^{n\times p}$ *and* $\mathbf{X}_2 \in \mathbb{R}^{n\times q}$, *let* $\mathbf{X} = (\mathbf{X}_1 : \mathbf{X}_2)$, *and let* $\mathbf{y}$ *be an* $n \times 1$ *random vector with dispersion matrix proportional to* $\mathbf{I}_n$. *Further, let* $\mathbf{F} = \mathbf{X}\mathbf{V}^{(1)}$ *and* $\mathbf{G} = \mathbf{X}\mathbf{V}^{(2)}$, *where* $(\mathbf{V}^{(1)} : \mathbf{V}^{(2)})$ *is a generalized inverse of* $\mathbf{V}$ *defined in* (2.7). *Then the following four statements are equivalent*:

(a) *there are no unit canonical correlations between* $\mathbf{X}_1'\mathbf{y}$ *and* $\mathbf{X}_2'\mathbf{y}$;

(b) *either of the conditions* $\mathbf{X}_1\mathbf{F}'\mathbf{X}_1 = \mathbf{X}_1$ *and* $\mathbf{X}_2\mathbf{G}'\mathbf{X}_1 = \mathbf{0}$ *holds along with either of the conditions* $\mathbf{X}_1\mathbf{F}'\mathbf{X}_2 = \mathbf{0}$ *and* $\mathbf{X}_2\mathbf{G}'\mathbf{X}_2 = \mathbf{X}_2$;

(c) *all the four conditions*

$$\mathbf{X}_1\mathbf{F}'\mathbf{X}_1 = \mathbf{X}_1, \qquad \mathbf{X}_2\mathbf{G}'\mathbf{X}_1 = \mathbf{0}, \qquad \mathbf{X}_1\mathbf{F}'\mathbf{X}_2 = \mathbf{0}, \qquad \mathbf{X}_2\mathbf{G}'\mathbf{X}_2 = \mathbf{X}_2 \quad (3.1)$$

hold simultaneously,

(d) $\mathbf{X}_1\mathbf{F}' = \mathbf{P}_{1\cdot 2}$, $\mathbf{X}_2\mathbf{G}' = \mathbf{P}_{2\cdot 1}$.

Moreover, if the conditions above are satisfied and the elements of the sets $\{\mathrm{cc}(\mathbf{X}_1'\mathbf{y}, \mathbf{X}_2'\mathbf{y})\}$ *and* $\{\mathrm{cc}(\mathbf{F}'\mathbf{y}, \mathbf{G}'\mathbf{y})\}$ *are ordered decreasingly, then*

$$\mathrm{cc}_i(\mathbf{X}_1'\mathbf{y}, \mathbf{X}_2'\mathbf{y}) \geqslant \mathrm{cc}_{i+u_0}(\mathbf{F}'\mathbf{y}, \mathbf{G}'\mathbf{y}), \qquad i = 1, \ldots, t,$$

where $t = r(\mathbf{X}_1'\mathbf{X}_2) = \#\{\mathrm{cc}(\mathbf{X}_1'\mathbf{y}, \mathbf{X}_2'\mathbf{y})\}$, *and* $t + u_0 = r(\mathbf{F}'\mathbf{G}) = \#\{\mathrm{cc}(\mathbf{F}'\mathbf{y}, \mathbf{G}'\mathbf{y})\}$, *with* $u_0 = \dim \mathscr{C}(\mathbf{F}) \cap \mathscr{C}(\mathbf{G})$ *being the number of unit canonical correlations between* $\mathbf{F}'\mathbf{y}$ *and* $\mathbf{G}'\mathbf{y}$.

Proof. From $\mathbf{P}_{\mathbf{X}} = \mathbf{X}_1\mathbf{F}' + \mathbf{X}_2\mathbf{G}'$ [cf. Lemma 4(c)] it follows that $(\mathbf{X}_1\mathbf{F}' + \mathbf{X}_2\mathbf{G}')\mathbf{X} = \mathbf{X}$. Moreover, from (2.9) and (2.10) it is seen that $\mathscr{C}(\mathbf{F}\mathbf{X}_1') \subseteq \mathscr{C}(\mathbf{X})$ and $\mathscr{C}(\mathbf{G}\mathbf{X}_2') \subseteq \mathscr{C}(\mathbf{X})$. Consequently, the result follows by applying Lemma 5(b) and Lemma 2 with $\mathbf{A} = \mathbf{X}_1$, $\mathbf{B} = \mathbf{X}_2$, $\mathbf{C} = \mathbf{F}$, and $\mathbf{D} = \mathbf{G}$.

For the proof of the second part notice that the conditions (d) imply that $\mathscr{C}(\mathbf{F}\mathbf{X}_1') = \mathscr{C}(\mathbf{Q}_2\mathbf{X}_1)$ and $\mathscr{C}(\mathbf{G}\mathbf{X}_2') = \mathscr{C}(\mathbf{Q}_1\mathbf{X}_2)$. Consequently, since property (b) in Lemma 3 is equivalent to

$$\mathscr{C}(\mathbf{F}) \boxplus \mathscr{C}(\mathbf{Q}_{\mathbf{F}}\mathbf{G}) = \mathscr{C}(\mathbf{X}_2) \boxplus \mathscr{C}(\mathbf{Q}_2\mathbf{X}_1)$$

and

$$\mathscr{C}(\mathbf{G}) \boxplus \mathscr{C}(\mathbf{Q}_{\mathbf{G}}\mathbf{F}) = \mathscr{C}(\mathbf{X}_1) \boxplus \mathscr{C}(\mathbf{Q}_1\mathbf{X}_2),$$

it follows that $\mathscr{C}(\mathbf{Q}_{\mathbf{F}}\mathbf{G}) \subseteq \mathscr{C}(\mathbf{X}_2)$ and $\mathscr{C}(\mathbf{Q}_{\mathbf{G}}\mathbf{F}) \subseteq \mathscr{C}(\mathbf{X}_1)$. Then Lemmas 4 and 5 imply that

$$\begin{aligned}\mathrm{cc}_i^2(\mathbf{X}_1'\mathbf{y}, \mathbf{X}_2'\mathbf{y}) &= \mathrm{ch}_i(\mathbf{P}_1\mathbf{P}_2)\\ &\geqslant \mathrm{ch}_i(\mathbf{P}_{\mathbf{Q}_{\mathbf{F}}\mathbf{G}}\mathbf{P}_{\mathbf{Q}_{\mathbf{G}}\mathbf{F}}) = \mathrm{cc}_{i+u_0}^2(\mathbf{F}'\mathbf{y}, \mathbf{G}'\mathbf{y}), \qquad i = 1, \dots, t,\end{aligned}$$

thus concluding the proof. ∎

THEOREM 2. *Let* $\mathbf{X}_1 \in \mathbb{R}^{n\times p}$ *and* $\mathbf{X}_2 \in \mathbb{R}^{n\times q}$, *let* $\mathbf{X} = (\mathbf{X}_1 : \mathbf{X}_2)$, *and let* $\mathbf{y}$ *be an* $n \times 1$ *random vector with the dispersion matrix proportional to* $\mathbf{I}_n$. *Further, let* $\mathbf{F} = \mathbf{X}\mathbf{V}^{(1)}$ *and* $\mathbf{G} = \mathbf{X}\mathbf{V}^{(2)}$, *where* $(\mathbf{V}^{(1)} : \mathbf{V}^{(2)})$ *is a generalized inverse of* $\mathbf{V}$ *defined in* (2.7). *Then the following four statements are equivalent*:

(a) *there are no unit canonical correlations between* $\mathbf{F}'\mathbf{y}$ *and* $\mathbf{G}'\mathbf{y}$;

(b) *either of the conditions* $\mathbf{F}\mathbf{X}_1'\mathbf{F} = \mathbf{F}$ *and* $\mathbf{G}\mathbf{X}_2'\mathbf{F} = \mathbf{0}$ *holds along with either of the conditions* $\mathbf{F}\mathbf{X}_1'\mathbf{G} = \mathbf{0}$ *and* $\mathbf{G}\mathbf{X}_2'\mathbf{G} = \mathbf{G}$;

(c) *all the four conditions*

$$\mathbf{F}\mathbf{X}_1'\mathbf{F} = \mathbf{F}, \qquad \mathbf{G}\mathbf{X}_2'\mathbf{F} = \mathbf{0}, \qquad \mathbf{F}\mathbf{X}_1'\mathbf{G} = \mathbf{0}, \qquad \mathbf{G}\mathbf{X}_2'\mathbf{G} = \mathbf{G} \quad (3.2)$$

hold simultaneously;

(d) $\mathbf{F}\mathbf{X}_1' = \mathbf{P}_{\mathbf{F}\cdot\mathbf{G}}$, $\mathbf{G}\mathbf{X}_2' = \mathbf{P}_{\mathbf{G}\cdot\mathbf{F}}$.

Moreover, if the conditions above are satisfied and the elements of the sets $\{\mathrm{cc}(\mathbf{X}_1'\mathbf{y}, \mathbf{X}_2'\mathbf{y})\}$ *and* $\{\mathrm{cc}(\mathbf{F}'\mathbf{y}, \mathbf{G}'\mathbf{y})\}$ *are ordered decreasingly, then*

$$\mathrm{cc}_i(\mathbf{F}'\mathbf{y}, \mathbf{G}'\mathbf{y}) \geqslant \mathrm{cc}_{i+u}(\mathbf{X}_1'\mathbf{y}, \mathbf{X}_2'\mathbf{y}), \qquad i = 1, \dots, t_0,$$

where $t_0 = \mathrm{r}(\mathbf{F}'\mathbf{G}) = \#\{\mathrm{cc}(\mathbf{F}'\mathbf{y}, \mathbf{G}'\mathbf{y})\}$, *and* $t_0 + u = \mathrm{r}(\mathbf{X}_1'\mathbf{X}_2) = \#\{\mathrm{cc}(\mathbf{X}_1'\mathbf{y}, \mathbf{X}_2'\mathbf{y})\}$, *with* $u = \dim \mathscr{C}(\mathbf{X}_1) \cap \mathscr{C}(\mathbf{X}_2)$ *being the number of unit canonical correlations between* $\mathbf{X}_1'\mathbf{y}$ *and* $\mathbf{X}_2'\mathbf{y}$.

Proof. From $\mathbf{P}_{(\mathbf{F}:\mathbf{G})} = \mathbf{F}\mathbf{X}_1' + \mathbf{G}\mathbf{X}_2'$ [cf. Lemma 3(c)] it follows that $(\mathbf{F}\mathbf{X}_1' + \mathbf{G}\mathbf{X}_2')(\mathbf{F} : \mathbf{G}) = (\mathbf{F} : \mathbf{G})$. Moreover, since $\mathscr{C}(\mathbf{X}) = \mathscr{C}(\mathbf{F} : \mathbf{G})$ [cf. Lemma 3(b)], it is clear that $\mathscr{C}(\mathbf{X}_1\mathbf{F}') \subseteq \mathscr{C}(\mathbf{F} : \mathbf{G})$ and $\mathscr{C}(\mathbf{X}_2\mathbf{G}') \subseteq \mathscr{C}(\mathbf{F} : \mathbf{G})$. Consequently, the result follows by applying Lemma 5(b) and Lemma 2 with $\mathbf{A} = \mathbf{F}$, $\mathbf{B} = \mathbf{G}$, $\mathbf{C} = \mathbf{X}_1$, and $\mathbf{D} = \mathbf{X}_2$. The proof of the second part is similar to the proof of the second part of Theorem 1. ■

The following theorem characterizes the situations where all the canonical correlations between $\mathbf{X}_1'\mathbf{y}$ and $\mathbf{X}_2'\mathbf{y}$ and all the canonical correlations between $\mathbf{F}'\mathbf{y}$ and $\mathbf{G}'\mathbf{y}$ are less than one simultaneously.

THEOREM 3. *Let* $\mathbf{X}_1 \in \mathbb{R}^{n\times p}$ *and* $\mathbf{X}_2 \in \mathbb{R}^{n\times q}$, *let* $\mathbf{X} = (\mathbf{X}_1 : \mathbf{X}_2)$, *and let* $\mathbf{y}$ *be an* $n \times 1$ *random vector with dispersion matrix proportional to* $\mathbf{I}_n$. *Further, let* $\mathbf{F} = \mathbf{X}\mathbf{V}^{(1)}$ *and* $\mathbf{G} = \mathbf{X}\mathbf{V}^{(2)}$, *where* $(\mathbf{V}^{(1)} : \mathbf{V}^{(2)})$ *is a generalized inverse of* $\mathbf{V}$ *defined in* (2.7). *Consider the following statements*:

(a) *all the canonical correlations between* $\mathbf{X}_1'\mathbf{y}$ *and* $\mathbf{X}_2'\mathbf{y}$ *are less than one*;

(b) *all the canonical correlations between* $\mathbf{F}'\mathbf{y}$ *and* $\mathbf{G}'\mathbf{y}$ *are less than one*;

(c_1) $\mathscr{C}(\mathbf{F}) = \mathscr{C}(\mathbf{Q}_2\mathbf{X}_1)$ *and* $\mathscr{C}(\mathbf{G}) = \mathscr{C}(\mathbf{Q}_1\mathbf{X}_2)$;

(c_2) $\mathbf{X}_1'\mathbf{G} = \mathbf{0}$ *and* $\mathbf{X}_2'\mathbf{F} = \mathbf{0}$;

(c_3) $\mathbf{X}_1'\mathbf{G} = \mathbf{0}$, $\mathrm{r}(\mathbf{X}_1) = \mathrm{r}(\mathbf{F})$, *and* $\mathrm{r}(\mathbf{X}_2) = \mathrm{r}(\mathbf{G})$;

(c_4) $\mathbf{X}_2'\mathbf{F} = \mathbf{0}$, $\mathrm{r}(\mathbf{X}_1) = \mathrm{r}(\mathbf{F})$, *and* $\mathrm{r}(\mathbf{X}_2) = \mathrm{r}(\mathbf{G})$;

(c_5)$\mathbf{P}_{\mathbf{F}\cdot\mathbf{G}} = \mathbf{P}_{1\cdot 2}'$ *and* $\mathbf{P}_{\mathbf{G}\cdot\mathbf{F}} = \mathbf{P}_{2\cdot 1}'$.

Then

$$[(\mathrm{a}) \text{ and } (\mathrm{b})] \Leftrightarrow (\mathrm{c}_1) \Leftrightarrow (\mathrm{c}_2) \Leftrightarrow (\mathrm{c}_3) \Leftrightarrow (\mathrm{c}_4) \Leftrightarrow (\mathrm{c}_5).$$

Moreover, if the conditions above are satisfied, then

$$\{\mathrm{cc}(\mathbf{X}_1'\mathbf{y}, \mathbf{X}_2'\mathbf{y})\} = \{\mathrm{cc}(\mathbf{F}'\mathbf{y}, \mathbf{G}'\mathbf{y})\},$$

where the cardinalities of these sets are $\mathrm{r}(\mathbf{X}_1'\mathbf{X}_2) = \mathrm{r}(\mathbf{F}'\mathbf{G})$.

Proof. From Theorems 1 and 2 it follows that if (a) and (b) hold, then

$$\mathscr{C}(\mathbf{F}) = \mathscr{C}(\mathbf{F}\mathbf{X}_1') = \mathscr{C}(\mathbf{P}_{1\cdot 2}') = \mathscr{C}(\mathbf{Q}_2\mathbf{X}_1)$$

and

$$\mathscr{C}(\mathbf{G}) = \mathscr{C}(\mathbf{G}\mathbf{X}_2') = \mathscr{C}(\mathbf{P}_{2\cdot 1}') = \mathscr{C}(\mathbf{Q}_1\mathbf{X}_2),$$

i.e., (a) and (b) imply (c_1). Premultiplying in (c_1) by $\mathbf{X}_2'$ and $\mathbf{X}_1'$, respectively, yields the two equalities in (c_2). From Theorems 1 and 2 it is clear that these equalities entail (a) and (b). From the same theorems it follows that if the conditions (c_2) hold, then $\mathbf{F}'$ and $\mathbf{G}'$ are reflexive generalized inverses of $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively, and hence $\mathrm{r}(\mathbf{X}_1) = \mathrm{r}(\mathbf{F})$ and $\mathrm{r}(\mathbf{X}_2) = \mathrm{r}(\mathbf{G})$. From Lemma 3 is is known that $\mathbf{P}_{\mathbf{X}} = \mathbf{F}\mathbf{X}_1' + \mathbf{G}\mathbf{X}_2' = \mathbf{P}_{(\mathbf{F}:\mathbf{G})}$, which shows that if the conditions (c_3) hold, then $\mathbf{F}$ and $\mathbf{G}$ are reflexive generalized inverses of $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively. Consequently, the equality of $\mathbf{G}\mathbf{X}_2'\mathbf{F} = \mathbf{0}$ may be strengthened to $\mathbf{X}_2'\mathbf{F} = \mathbf{0}$, thus establishing the equivalence between (c_2) and (c_3). The equivalence between (c_2) and (c_4) follows similarly, and the second part follows directly by combining the second parts of Theorems 1 and 2. From Theorems 1 and 2 it is clear that (a) and (b) imply (c_5). Therefore, to complete the proof it suffices to prove the relation (c_5) $\Rightarrow$ (a) & (b). Since $\mathrm{r}(\mathbf{P}_{1\cdot 2}') = \mathrm{r}(\mathbf{Q}_2\mathbf{X}_1)$ and $\mathscr{C}(\mathbf{P}_{1\cdot 2}') \subseteq \mathscr{C}(\mathbf{Q}_2\mathbf{X}_1)$ we have $\mathscr{C}(\mathbf{P}_{1\cdot 2}') = \mathscr{C}(\mathbf{Q}_2\mathbf{X}_1)$. If ($c_5$) holds, then we must also have $\mathscr{C}(\mathbf{P}_{\mathbf{F}\cdot\mathbf{G}}) = \mathscr{C}(\mathbf{Q}_2\mathbf{X}_1)$, and so $\mathscr{C}(\mathbf{P}_{\mathbf{F}\cdot\mathbf{G}})$ is invariant with respect to the choice of the generalized inverse appearing in the formula for $\mathbf{P}_{\mathbf{F}\cdot\mathbf{G}}$. In view of Lemma 1(b), this means that (b) holds. The part "(c_5) $\Rightarrow$ (a)" follows similarly. ■

We conclude by furnishing necessary and sufficient conditions which connect the information that there are no unit canonical correlations between $\mathbf{X}_1'\mathbf{y}$ and $\mathbf{X}_2'\mathbf{y}$ with the information that there are no unit canonical correlations between $\mathbf{F}'\mathbf{y}$ and $\mathbf{G}'\mathbf{y}$.

THEOREM 4. *Let* $\mathbf{X}_1 \in \mathbb{R}^{n\times p}$ *and* $\mathbf{X}_2 \in \mathbb{R}^{n\times q}$, *let* $\mathbf{X} = (\mathbf{X}_1 : \mathbf{X}_2)$, *and let* $\mathbf{y}$ *be an* $n \times 1$ *random vector with dispersion matrix proportional to* $\mathbf{I}_n$. *Further, let* $\mathbf{F} = \mathbf{X}\mathbf{V}^{(1)}$ *and* $\mathbf{G} = \mathbf{X}\mathbf{V}^{(2)}$, *where* $(\mathbf{V}^{(1)} : \mathbf{V}^{2)})$ *is a generalized* *inverse of* $\mathbf{V}$ *defined in* (3.1). *Consider the following statements:*

(a) *all the canonical correlations between* $\mathbf{X}_1'\mathbf{y}$ *and* $\mathbf{X}_2'\mathbf{y}$ *are less than one;*
(b) *all the canonical correlations between* $\mathbf{F}'\mathbf{y}$ *and* $\mathbf{G}'\mathbf{y}$ *are less than one;*
(c) $r(\mathbf{F}) = r(\mathbf{X}_1)$ *and* $r(\mathbf{G}) = r(\mathbf{X}_2)$.

Then any two of the conditions (a), (b), *and* (c) *imply the third condition.*

Proof. From Theorems 1 and 2 it clear that (a) and (b) imply (c). According to Theorem (1), (a) implies that $\mathbf{X}_1\mathbf{F}'\mathbf{X}_1 = \mathbf{X}_1$, which means that $\mathbf{F}'$ is a generalized inverse of $\mathbf{X}_1$. Combining this condition with $r(\mathbf{F}) = r(\mathbf{X}_1)$ implies that $\mathbf{F}'$ is a reflexive generalized inverse of $\mathbf{X}_1$, i.e., $\mathbf{F}\mathbf{X}_1'\mathbf{F} = \mathbf{F}$ [cf., e.g., Rao and Mitra (1971, Lemma 2.5.1)]. In view of Theorem 2, this establishes the part "(a) & (c) $\Rightarrow$ (b)." The part "(b) & (c) $\Rightarrow$ (a)" follows similarly. ■

Notice that condition (c) alone does not imply (a) and (b). An example is provided by the matrices

$$\mathbf{X}_1' = \mathbf{X}_2' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{V}^{(1)\prime} = \mathbf{V}^{(2)\prime} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

in which case neither (a) nor (b) of Theorem 4 holds.

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